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Electromagnetic Theory. (For Q-2)

Derivation of Maxwell's Field equns:—

(1) From Gauss's theorem in electrostatics the relation between electric field intensity vector $\vec{E}$ through a closed surface and the net charge Q enclosed within that surface can be written as

$$\int_S \vec{E}\cdot d\vec{S} = \frac{Q}{\epsilon_0} \quad \text{—— (1)}$$

where S is the total surface area. If this surface area enclosed a volume v then then transforming the surface integral in eqn (1) into volume integral by Gauss's divergence theorem, we get

$$\int_v \nabla\cdot\vec{E}\cdot dv = \frac{Q}{\epsilon_0} \quad \text{—— (2)}$$

If ρ' be the volume density of bound charge and ρ that of the free charges then we get

$$Q = \int_v (\rho+\rho')\, dv \quad \text{—— (3)}$$

Hence from (2) & (3) we get

$$\nabla\cdot\vec{E} = \frac{\rho+\rho'}{\epsilon_0} \quad \text{—— (4)}$$

at every point of the region. If $\vec{P}$ be the polarisation vector then

$$\rho' = -\vec{\nabla}\cdot\vec{P}$$

Hence eqn (4) becomes

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left(\rho - \vec{\nabla}\cdot\vec{P}\right)$$

$$\text{or } \epsilon_0 \vec{\nabla}\cdot\vec{E} = \rho - \vec{\nabla}\cdot\vec{P}$$

$$\text{or } \epsilon_0 \vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P} = \rho$$

$$\text{or } \vec{\nabla}\cdot\left(\epsilon_0\vec{E} + \vec{P}\right) = \rho$$

$$\text{or } \vec{\nabla}\cdot\vec{D} = \rho \quad \text{——— (5)}$$

where the vector $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, is called electric displacement vector.

(2) According to Biot-Savart law we have

$$\vec{B} = \frac{\mu_0 i}{4\pi} \oint \frac{\vec{dl} \times r_1}{r^2} \quad \text{——— (6)}$$

where $\vec{B}$ is the magnetic induction vector at a point in space due to a current loop, $\vec{dl}$ is an element of current i in the loop, $\vec{r}_1$ is a unit vector in the positive direction joining the current and the point under consideration, r is the distance between the current element and the point & μ_0 is permeability of free space. Taking divergence of reln. (6) we get

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0 i}{4\pi} \cdot \vec{\nabla} \oint \frac{\vec{dl} \times \vec{r}_1}{r^2}$$

Here we can interchange the order of the operators $\vec{\nabla}$ and $\oint$. Hence

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0 i}{4\pi} \cdot \oint \nabla \cdot \frac{(\vec{dl} \times \vec{r}_1)}{r^2}$$

$$= \frac{\mu_0 i}{4\pi} \oint \left[\frac{\vec{r}_1}{r^2} \cdot \vec{\nabla} \times \vec{dl} - \vec{dl}\left(\vec{\nabla} \cdot \frac{\vec{r}_1}{r^2}\right)\right]$$

$$\because \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A} - A\cdot\vec{\nabla}\times\vec{B}$$

Since $\vec{dl}$ is not a fn. of the co-ordinate x, y & z of the point P where we wish to calculate

$$\vec{\nabla}\cdot\vec{B} \quad so \quad \vec{\nabla}\times\vec{dl} = 0$$

Further $\qquad \vec{\nabla}\times\left(\frac{\vec{r_1}}{r^2}\right) = -\vec{\nabla}\times\vec{\nabla}\left(\frac{1}{r}\right)$,

because $\quad \vec{\nabla}\left(\frac{1}{r}\right) = -\frac{r}{r^2} = 0$

curl grad $\phi = 0 \quad$ so, $\vec{\nabla}\cdot\vec{B} = 0$

(3) According to Faraday's law of electromagnetic induction we know that

$$\oint \vec{E}\cdot\vec{dl} = -\frac{d\phi}{dt} \quad -(7)$$

where ϕ is the magnetic flux linked with the circuit and is given by

$$\phi = \int B\cdot da \ ; \text{ where s in any surface bounded by the circuit.}$$

Hence eqn (7) becomes

$$\oint \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int \vec{B}\cdot\vec{da}$$

Transforming the line integral into surface integral with the help of stoke's theorem we get

$$\int_S (\vec{\nabla}\times\vec{E})\cdot\vec{da} = -\frac{d}{dt}\left(\int_S \vec{B}\cdot\vec{da}\right)$$

Since path of integration is fixed in space and hence the order of differentiation & integration can be interchanged on the right hand side. So,

$$\int_S (\vec{\nabla}\times\vec{E})\cdot\vec{da} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{da}$$

~~potential derivative of $\vec{B}$ has been used~~

because we need rate of change of $\vec{B}$ with time at a fixed point. Since the above eqn. is valid for arbitrary surfaces we get

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} = -\vec{\dot{B}}$$

$$\therefore \vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$$

4) From Ampere's law we know that

$$\oint \vec{H} \cdot \vec{dl} = \int \vec{J} \cdot \vec{da} \quad -(1)$$

where $\vec{J}$ is conduction current. Using Stoke's theorem this can be written as

$$\int \vec{\nabla} \times \vec{H} \cdot \vec{da} = \int \vec{J} \cdot \vec{da}$$

Hence we get $\vec{\nabla} \times \vec{H} = \vec{J} \quad -(2)$

If we take divergence, eqn (2) will be

$$\text{div}(\text{curl}\,\vec{H}) = \text{div}\,\vec{J}$$

but div of curl of a vector is zero. Hence Ampere's theorem is equivalent to the result

$$\vec{\nabla} \cdot \vec{J} = 0$$

But from the eqn. of continuity we know that

$$\vec{\nabla} \cdot \vec{J} = \cancel{0} - \frac{\partial \rho}{\partial t} \quad -(3)$$

So, $\vec{\nabla} \cdot \vec{J} = 0$ leads to the condition that charge density ρ is independent of time. Hence we conclude that Ampere's law is correct for steady currents only and is not consistent with time varying eqn of continuity.

Using Maxwell's first law

ie $\vec{\nabla}\cdot\vec{D} = \rho$ in eqn (3) we get

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})$$

Interchanging the differentiation w.r. to space and time we get

$$\nabla\cdot\left(\frac{\partial \vec{D}}{\partial t} + \vec{J}\right) = 0 \quad \text{— (4)}$$

This is the ~~case~~ of form of eqn. of continuity in case of a time varying field. From eqn (4) we conclude that $\left(\frac{\partial \vec{D}}{\partial t} + \vec{J}\right)$ should be regarded as the total current density for time varying field. Since $\vec{D}$ is the displacement density, $\frac{\partial D}{\partial t}$ is known as the displacement current density.

In case of charge or discharge of a capacitor the conduction current in the wire attached to the plate is equal to the displacement current passing between the plates. Displacement current has heating, magnetic and other properties just like conduction current.

Thus recognising $\frac{\partial \vec{D}}{\partial t} + \vec{J}$ as the total current density in case of charging field eqn

$$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad ; \text{ which is}$$

Maxwell's 4th eqn.