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Mechanics : Motion under central forces

Kepler's laws

Q:- What are Kepler's laws of planetary motion? Give their mathematical derivation.

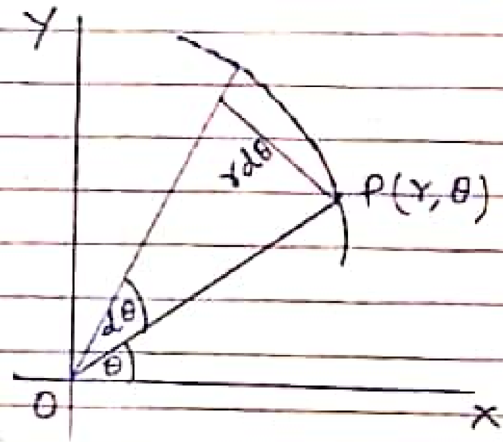
Ans:- The German astronomer Johaness Kepler gave three laws of planetary motion which are as follows.

(1). Each planet moves in an ellipse with the sun at one focus.

(2). The line joining a planet to the sun traces out equal areas in equal times, i.e the areal velocity of the radius vector is constant.

(3) The square of the time for the completion of one circuit of the orbit (the planet's year) is proportional to the cube of the semi major axis of the orbit.

Derivation : Let us chose the x, y axes in the plane of motion with O as origin. Let r & θ be the plane polar co-ordinates of a point where a particle of mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ is situated. Now the radial and transverse accelerations are given by



$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = \frac{rd^2\theta}{dt^2} + \frac{2dr}{dt}\cdot\frac{d\theta}{dt}$$

$$= r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

Hence, the equs. of motion along r and θ directions are

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = F(r) \quad \text{—— (1)}$$

and

$$\mu r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \text{—— (2)}$$

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Eqn. (2) can also be written as

$$\frac{d}{dt}(\mu r^2 \dot{\theta}) = 0$$

or $\quad \mu r^2 \dot{\theta} = \text{Constant}$

But $\quad \mu r^2 \dot{\theta} = L$, the angular momentum

Hence we have $\quad L = \mu r^2 \dot{\theta}^2 = \text{Constant}$ ——(3)

From the fig. the area swept out by the radius in time dt is

$$dA = \frac{1}{2} r \cdot r d\theta = \frac{1}{2} r^2 d\theta$$

Hence, the rate at which the area is swept out is given by

$$\dot{A} = \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2\mu} = \text{constant}$$

This is Kepler's second law.

The total energy ~~is~~ E is given by

$$E = T + V = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 + V(r)$$

$$= \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) \quad \text{——(4)}$$

From eqn. (4)

$$\dot{r} = \frac{dr}{dt}\left[\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}$$

$$\therefore dr = \left[\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)\right]^{1/2} dt \quad \text{——(5)}$$

From eqn. (3) $\quad d\theta = \frac{L}{\mu r^2} dt \quad$ ——— (6)

From eqn. (5) and (6) we have

$$d\theta = \frac{L/\mu r^2 \cdot dr}{\left[\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}}$$

$$= \frac{\frac{L}{\mu r^2} \cdot dr}{\left[\frac{2}{\mu}\left(E + \frac{k}{r} - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}} \quad \text{——(7)}$$

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Integrating we have

$$\theta = \int \frac{L/\mu r^2 . dr}{\left[\frac{2}{\mu}\left(E + \frac{K}{r} - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}} + \theta_0 \quad — (8)$$

Putting $r = \frac{1}{u}$, we have $dr = -\frac{du}{u^2}$

or $dr . u^2 = -du$

or $dr . \frac{1}{r^2} = -du$

$$\therefore \theta = \theta_0 - \int \frac{L . du}{\left[2\mu\left(E + \frac{K}{r} - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}}$$

$$= \theta_0 - \int \frac{L . du}{\left[2\mu E + 2\mu K u - L^2 u^2\right]^{1/2}}$$

$$= \theta_0 - \int \frac{du}{\left[\frac{2\mu E}{L^2} + \frac{2\mu K u}{L^2} - u^2\right]^{1/2}}$$

$$= \theta_0 - \int \frac{du}{\left[\sqrt{\left(\frac{2\mu E}{L^2} + \frac{2\mu K E}{L^4}\right)^2} - \left(u - \frac{\mu K}{L^2}\right)^2\right]^{1/2}}$$

$$= \theta_0 - \int \frac{du}{\left[\sqrt{\left(\frac{2\mu E}{L^2} + \frac{\mu^2 K^2}{L^4}\right)^2} - \left(u - \frac{\mu K}{L^2}\right)^2\right]^{1/2}}$$

The integral is of the form $\int \frac{dx}{\sqrt{a^2 - x^2}}$, which is standard equal to $-\cos^{-1}\frac{x}{a}$. Hence, we get

$$\theta = \theta_0 + \cos^{-1} \frac{u - \frac{\mu K}{L^2}}{\left[\frac{2\mu E}{L^2} + \frac{\mu^2 K^2}{L^4}\right]^{1/2}}$$

or, $$\cos(\theta - \theta_0) = \frac{u - \frac{\mu K}{L^2}}{\left[\frac{2\mu E}{L^2} + \frac{\mu^2 K^2}{L^4}\right]^{1/2}}$$

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Dividing by $\mu k/L^2$ we have

$$\cos(\theta-\theta_0) = \frac{\left(\frac{uL^2}{\mu k} - 1\right)}{\left[\frac{2EL^2}{\mu k^2} + 1\right]^{1/2}} \quad \text{--- (9)}$$

or $$\frac{uL^2}{\mu k} = 1 + \left(1 + \frac{2EL^2}{\mu k^2}\right)^{1/2} \cos(\theta-\theta_0)$$

or $$u = \frac{\mu k}{L^2}\left[1 + \left(1+\frac{2EL^2}{\mu k^2}\right)^{1/2} \cos(\theta-\theta_0)\right]$$

or $$\frac{1}{r} = C\left[1 + \epsilon\cos(\theta-\theta_0)\right] \quad \text{--- (10)}$$

where
$$C = \frac{\mu k}{L^2} \quad \text{and} \quad \left[1 + \frac{2EL^2}{\mu k^2}\right]^{1/2} = \epsilon$$

where ϵ = Eccentricity

which is the eqn. of a conic section with origin at the focus. The nature of the conic section depends on the value of ϵ

(i) $\epsilon < 1$, $E < 0$; Ellipse

(ii) $\epsilon > 1$, $E > 0$; Hyperbola

(iii) $\epsilon = 1$, $E = 0$; Parabola

(iv) $\epsilon = 0$, $E = -\frac{\mu k^2}{2L^2}$; circle

In the case of elliptic orbit when $\theta - \theta_0 = 0$, $r = r_1 =$ Perihelion (minimum distance of the planet from the sun.) when $\theta-\theta_0 = 180$, $r = r_2 =$ Aphelion (Max^m distance)

From eqn. (10), we have
$$r_1 = \frac{1}{C(1+\epsilon)} \quad \text{and} \quad r_2 = \frac{1}{C(1-\epsilon)}$$

$$a = \frac{r_1+r_2}{2} = \frac{1}{2C}\left[\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right] = \frac{1}{C(1-\epsilon^2)} = -\frac{k}{2E}$$

or $$E = \frac{-k}{2a}$$

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The total energy depends solely on semi major axis. Since the energy is -ve and $\epsilon = \sqrt{1 - \frac{2EL^2}{\mu k^2}}$ is less than unity, the conic section is an ellipse which establishes Kepler's first law.

We know that the period ~~is given~~ of revolution is given by

$$T = \frac{\text{Area of the elliptical orbit}}{\text{Areal velocity}} = \frac{\pi ab}{L/2\mu} = \frac{2\mu\pi ab}{L}$$

$$\text{But } b = a\sqrt{1-\epsilon^2} = a\sqrt{1-1-\frac{2EL^2}{\mu k^2}}$$

$$= a\sqrt{\left(\frac{-2EL^2}{\mu k^2}\right)} = \sqrt{\frac{aL^2}{\mu k}}$$

$$\therefore \quad T^2 = \frac{4\mu^2\pi^2a^2b^2}{L^2} = \frac{4\mu^2\pi^2a^2}{L^2} \times \frac{aL^2}{\mu k}$$

$$= \left(\frac{4\pi^2\mu}{k}\right) \cdot a^3$$

where

$$\frac{4\pi^2\mu}{k} = \text{Constant}$$

$$\therefore \quad T^2 \propto a^3$$

which is Kepler's 3rd law.

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