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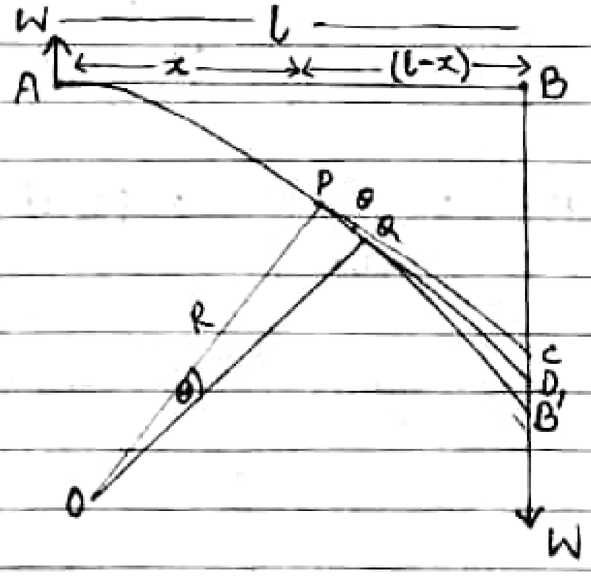
Bending of beam & Cantilever.
(For D-1 (H) & D-1 (Subs.)

Q:- Deduce an expression for the depression of a uniform bar held horizontally at one end and loaded at the other.

Ans:- 'A cantilever' is a beam fixed horizontally at one end and loaded at the other.

Let AB represent the neutral axis of a cantilever, of length L, fixed at the end A and loaded at B with a weight W, such that the end B is deflected into the position B' and the ~~axis~~ neutral axis takes up the position AB', it being assumed that the weight of the beam itself produces no bending.



Let us consider a section P of the beam at a distance x from the fixed end A. The normal moment of the external couple at this section, due to the ~~end~~ load W or the bending moment acting on it $= W \times PB' = W(l-x)$

Since the beam is in equilibrium, this must be equal to $\frac{YI}{R}$, where R is the radius of curvature of the neutral axis at P.

Therefore $W(l-x) = \frac{YI}{R}$

$$\text{or } R = \frac{YI}{W(l-x)} \quad \text{------} \quad (1)$$

This expression shows that the radius of curvature at a point of the beam varies inversely as $(l-x)$, the distance of the point from the loaded end. For a point Q, however at a small distance δx from P, it is practically the same as at P.

Let us draw tangents to the curve at P and Q, and let O be the centre and R, the radius of curvature of the portion PQ of the curve. Then if $\angle POQ = \theta$, we have

$$PQ = \delta x = R\theta \quad \text{------} \quad (ii)$$

Now, θ = difference in slope of the tangents at P & Q. And since the slope of the tangents at a point is measured by dy/dx at the point, we have

$$\therefore \theta = \left[\frac{dy}{dx} \text{ at } Q - \frac{dy}{dx} \text{ at } P\right]$$

Now the rate of change of slope is given by d^2y/dx^2.

$\therefore$ Change in slope from P to Q $= \delta x \cdot \frac{d^2y}{dx^2}$

$$\text{ie. } \theta = \delta x \cdot \frac{d^2y}{dx^2}$$

$$\text{or } \frac{\delta x}{R} = \delta x \cdot \frac{d^2y}{dx^2}$$

$$\text{or } \frac{1}{R} = \frac{d^2y}{dx^2} \quad \text{------} \quad (iii)$$

Since in the case of bent rods, or beams, the curve of the neutral axis is very slight, the relation

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$ gives the radius of curvature of the axis at any given point.

Now from eqn (1) we have

$$W(l-x) = YI/R,$$

the axis of x being taken along the horizontal and the axis of y, vertically downwards.
Substituting the value of $1/R$, we have

$$W(l-x) = Y.I.\frac{d^2y}{dx^2}$$

or $$\frac{YI}{W}.\frac{d^2y}{dx^2} = (l-x)$$

integrating this, we have

$$\int \frac{YI}{W}.\frac{d^2y}{dx^2}\,dx = \int (l-x)\,dx$$

or $$\frac{YI}{W}.\frac{dy}{dx} = \cancel{(l-x)^2}\left(lx - \frac{x^2}{2}\right) + C_1 \quad \text{— (iv)}$$

Clearly, $\frac{dy}{dx}$ is zero at A, ie $\frac{dy}{dx} = 0$ when $x = 0$

Substituting these values of dy/dx and x in eqn (iv) we have $C_1 = 0$

$$\therefore \frac{YI}{W}.\frac{dy}{dx} = \left(lx - x^2/2\right) \quad \text{—— (v)}$$

Again integrating we have

$$\int \frac{YI}{W}.\frac{dy}{dx}.dx = \int \left(lx - \frac{x^2}{2}\right)dx$$

or $$\frac{YI}{W}.y = \frac{lx^2}{2} - \frac{x^3}{6} + C_2 \quad \text{— (vi)}$$

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To determine c_2, we ~~have~~ see that the depression y of the rod is zero at the end A; so that $y = 0$ when $x = 0$. Putting ~~this~~ these values of x in eqn (vi) we have $c_2 = 0$

~~Hence~~

$$\cancel{\frac{YI}{W} \cdot y = \frac{lx^2}{2} - \frac{l^3}{6} = \frac{l^3}{3}}$$

$$\cancel{\text{or } y = \frac{Wl^3}{3YI} \quad \text{—— (vii)}}$$

Hence

$$\frac{YI}{W} \cdot y = \frac{lx^2}{2} - \frac{x^3}{6} \quad \text{—— (vii)}$$

Now to obtain the deflection of the loaded end, let us put $x = l$. Then clearly

$$\frac{YI}{W} \cdot y = \frac{l^3}{2} - \frac{l^3}{6} = \frac{l^3}{3}$$

$$\text{or} \quad y = \frac{Wl^3}{3YI} \quad \text{—— (viii)}$$

For the rod of the rectangular cross-section

$I = \frac{bd^3}{12}$ where b and d are breadth & depth.

So that for a rectangular Rod

$$y = \frac{Wl^3}{3Y} \cdot \frac{12}{bd^3} = \frac{4Wl^3}{Ybd^3}$$

For cylindrical rod $I = \frac{\pi r^4}{4}$, so for cylindrical rod,

$$y = \frac{Wl^3}{3YI} = \frac{Wl^3}{3Y} \cdot \frac{4}{\pi r^4}$$

$$= \frac{4Wl^3}{3Y\pi r^4}.$$