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E.M. Theory [Continued --]

D-2 (H)

Equation of electromagnetic wave in a isotropic, homogeneous & dielectric medium.

In case of a dielectric medium we know that conduction current density $\vec{J} = 0$ and further the volume density of charge $\rho = 0$. Hence Maxwell's equ^n reduce to

$$\vec{\nabla} \cdot \vec{D} = 0 \quad \text{--- (1)}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{--- (2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (3)}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad \text{--- (4)}$$

We know that for a pure dielectric medium $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ where ϵ and μ are permitivity & permeability of the medium under considerations.

Now from Maxwell's 3rd equ^n, we have

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (\because B = \mu_0 H)$$

or

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

Now taking curl on both sides of this equ^n, we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\mu_0 \frac{\partial \vec{H}}{\partial t}\right)$$

$$= -\mu_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H})$$

But from equ^n (4)

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\therefore \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu\frac{\partial}{\partial t}\left(\epsilon\frac{\partial\vec{E}}{\partial t}\right)$$
$$= -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{— (5)}$$

Now the curl of a curl of a vector function is

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{E}$$
$$= \vec{\nabla}\vec{\nabla}\cdot\vec{E} - \vec{\nabla}^2\vec{E}$$

for a charge free region, $\vec{\nabla}\cdot\vec{E} = 0$,

$$\therefore -\vec{\nabla}^2\vec{E} = -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{— (6)}$$

Since wave propagates along x axis and hence it is independent of y & z axes. Thus $\vec{\nabla}^2\vec{E}$ can be replace as

$$\frac{\partial^2\vec{E}}{\partial x^2} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{— (7)}$$

Similarly taking curl of equ^n (4) on the both sides we have

$$\frac{\partial^2\vec{H}}{\partial x^2} = \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

Now from acoustics, comparing it with the standard diff equ^n of wave motion

$$\frac{d^2y}{dx^2} = \frac{1}{c^2}\frac{d^2y}{dt^2} \quad \text{— (8)}$$

$\therefore$ from (7) & (8)

$$\frac{1}{c^2} = \mu\epsilon \quad \text{or} \quad c^2 = \frac{1}{\mu\epsilon}$$

For vaccum $\mu \to \mu_0$, $\epsilon \to \epsilon_0$ $\therefore c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8$ m/sec = velocity of li

Poynting Theorem :

Energy density of e.m field and Poynting theorem.
or The energy conservation law of the e.m field or
or Explain the significance of Poynting vector.
or Show that the flow of an e.m ~~fe~~ field energy at point is described by Poynting vector.
or Explain Poynting vector. Evaluate its magnitude for a plane electromagnetic wave in an isotropic medium.

Ans:- We know the general expression for the electrostatic and magnetostatic field energies are.

$$\left.\begin{aligned} \omega_e &= \frac{1}{2}\int_V (\vec{E}\cdot\vec{D})\,dv \\ \omega_m &= \frac{1}{2}\int_V (\vec{H}\cdot\vec{B})\,dv \end{aligned}\right\} \quad \text{--- (1)}$$

Now we will find the expression for the electromagnetic field energy in time dependent situation.

To obtain the expression for e.m field energy (or energy conservation law) we write III & IV Maxwell's field equ^ns

$$\text{Curl }\vec{E} = -\frac{\partial B}{\partial t} \quad \text{--- (2)}$$

$$\& \text{ Curl }\vec{H} = J + \frac{\partial \vec{D}}{\partial t} \quad \text{--- (3)}$$

Taking scalar product of equ^n (2) by $\vec{H}$ and that of (3) by $\vec{E}$ and subtracting ~~substituting~~ (2) from (3) we get

$$\vec{E}\cdot\text{Curl }\vec{H} - \vec{H}\cdot\text{Curl }\vec{E} = \vec{J}\cdot\vec{E} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}$$

and using $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\,\text{curl}\,\vec{A} - \vec{A}\cdot\text{curl}\,\vec{B}$

we write

$$\text{div}(\vec{H}\times\vec{E}) = \vec{J}\cdot\vec{E} + \vec{E}\,\frac{\partial D}{\partial t} + H\cdot\frac{\partial \vec{B}}{\partial t}$$

Taking volume integral over an arbitrary volume enclosed by the surface S on both sides we get

$$\int_V \text{div}(\vec{H}\times\vec{E})\,dv = \int_V (\vec{J}\cdot\vec{E})\,dv + \int_V \left[\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right] dv$$

or $$\int_V \left[\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) + \left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)\right] dv = -\int_V (\vec{J}\cdot\vec{E})\,dv - \int_V \text{div}(\vec{E}\times\vec{H})\,dv$$

Converting the 2nd terms on R.H.S into surface integral using Gauss's theorem, we have

$$\int_V \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + H\cdot\frac{\partial\vec{B}}{\partial t}\right) dv = -\int_V (J\cdot E)\,dv - \int_S (\vec{E}\times\vec{H})\,d\vec{S} \quad \text{—— (4)}$$

For an isotropic medium

$$\vec{D} = \epsilon\,\vec{E} \quad \text{and} \quad \vec{B} = \mu\,\vec{H}$$

We have
$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \vec{E}\cdot\frac{\partial}{\partial t}(\epsilon\vec{E})$$
$$= \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\epsilon\vec{E}) = \frac{1}{2}\frac{\partial}{\partial t}(\epsilon E^2)$$
$$= \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D}) = w_E \quad \text{—— (4a)}$$

Similarly $$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\mu H^2) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{B}) = w_m \quad \text{—— (4b)}$$

Putting these values ~~eqn~~ (4a) & (4b) in (4) and changing the order of integration and differentiation we get

$$\frac{\partial}{\partial t}\int_V \frac{(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})}{2}\,dv = -\int_V (\vec{J}\cdot\vec{E})\,dv - \int_S (\vec{E}\times\vec{H})\,d\vec{s} \quad \text{—— (5)}$$

Here $\frac{\vec{E}\cdot\vec{D}}{2} = \frac{1}{2}\epsilon E^2 = w_e$ = Electrical energy density

$\therefore \int \left(\frac{\vec{E}\cdot\vec{D}}{2}\right) dv = \omega_e$ = elec. energy.

and $\frac{\vec{H}\cdot\vec{B}}{2} = \frac{1}{2}\mu H^2 = w_m$ = mag. energy density

and $\int \frac{\vec{H}\cdot\vec{B}}{2}\,dv = \omega_m$ = mag. energy.

$$\therefore \int_V \left(\frac{\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}}{2}\right) dv = \omega_e + \omega_m = \omega \quad \text{—— (5a)}$$

ie Total energy of elec. & mag. field contained within the volume (V).

Also $\vec{J}$ = el. current density $= \rho.\vec{v}$

where ρ = volume density of charge

$\vec{v}$ = charge velocity

$\therefore \int (\vec{J}\cdot\vec{E})\,dv := \int (\rho\vec{v}\cdot\vec{E})\,dv = \frac{\partial T}{\partial t}$ = Rate of change of K.E of the charged bodies embeded in the field. —— (5b)

Equn (5b) Can be explained as follows

$\because \rho dv = dq$ = charge element

and $\vec{E}dq$ = force acting on the charge element

$\therefore \vec{E}dq\cdot d\vec{r}$ = work done by the e.m. force on charge element in moving it by $d\vec{r}$.

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$$\therefore \vec{E}dq\cdot\frac{d\vec{r}}{dt} = E\cdot dq\vec{v}$$
= work done in unit time.

$$\therefore \int \vec{J}\cdot E\cdot dv = \int_V \rho\vec{v}\,E\,dv = \int_V E\cdot dq\vec{v}$$
= Total charge work done in unit time.
$= \frac{\partial T}{\partial t}$ = Increase in K.E per unit time.

Thus putting (5a) & (5b) in equ^n (5) we get

$$\frac{\partial w}{\partial t} + \frac{\partial T}{\partial t} = -\int_S (\vec{E}\times\vec{H})\cdot ds \quad \text{—— (6)}$$

Change in total energy of the em field & charged material bodies per unit ~~volume~~ time.

Flux of field energy crossing per sec from S surface enclosing the volume V.

$\therefore \ \vec{E}\times\vec{H} = \vec{S}$ = Energy density flux vector, is called

$\vec{S}$ is called Poynting vector, its direction gives the direction of flow of energy. Thus the poynting vector is directed at right angles to the plane of $\vec{E}$ & $\vec{H}$. The poynting theorem can be stated as - " The decrease in energy in unit time, of an el. mag. field & the material charged bodies contained is equal to the energy density flux vector ($\vec{E}\times\vec{H}$) across the surface bounding ~~a~~ the field.

This is true when the surface is finite.

Case I: If the surface enclosing the field is infinitely distant then the field is zero at infinity, i.e $\vec{E}$ & $\vec{H}$ vanish at the boundary of the surface, then

$\int(\vec{E}\times\vec{H}).ds = 0$, and equ^n (6) reduces to

$$\frac{\partial w}{\partial t} + \frac{\partial T}{\partial t} = 0$$

$$\text{or } \frac{\partial}{\partial t}(w+T) = 0$$

$\therefore (w+T)$ = Total energy of the field = a constant.

Thus the total energy of el. m. field is conserved. This is known as energy conservation law of e.m field.

Hence ~~so~~ for an isolated system, the increase of w per unit time is due to the work done by the field on the material charged body in the field during this time.

Therefore an e.m field in vaccum can be regarded as a ~~real~~ mechanical system.

Case II :

In a non conducting medium ($T=0$ as $\vec{J}=0$) the energy law can be written ~~as~~ in the form of a hydrodynamical continuity equ^n for non compressible fluids

$$\frac{dw}{\partial t} + \text{div}\,\vec{S} = 0 \qquad \text{— (8a)}$$

Here in optics, the poynting vector ($\vec{S}$) is a measure of light intensity, and its direction represents the direction of propagation of light.

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