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Q:- Deduce Einstein's mass-energy relation $E = mc^2$, considering the variation of mass with velocity.

Ans - As we know, the kinetic energy of a body is measured by the amount of workdone to accelerate the body from its state of rest to the required state of motion at velocity v. If a force F acting on a body displaces it through a distance dx along its direction of application, the workdone by the force $= F \cdot dx$, which is stored by the body as its kinetic energy dE_k. Thus,

$$dE_k = F \cdot dx \quad \text{——} \quad (1)$$

From Newton second law of motion the force equals to the rate of change of momentum, i.e

$$F = \frac{d}{dt}(mv)$$

Since both m & v are variable

$$F = m\frac{dv}{dt} + v\frac{dm}{dt} \quad \text{——} \quad (2)$$

$$\therefore F \cdot dx = \left(m \cdot \frac{dv}{dt} + v\frac{dm}{dt}\right) \cdot dx$$

$$\therefore E_{dk} = \left(m \cdot \frac{dv}{dt} + v\frac{dm}{dt}\right) \cdot dx$$

$$= m \cdot \frac{dv}{dt} \cdot dx + v\frac{dm}{dt} \cdot dx$$

$$= m \cdot \frac{dx}{dt} \cdot dv + v \cdot \frac{dx}{dt} \cdot dm$$

$$= m \cdot v\, dv + v^2 \cdot dm \quad \text{——} \quad (3)$$

Now according to mass transformation eqn.

$$m = m_0 \Big/ \sqrt{1 - v^2/c^2}$$

or $$m^2 = \frac{m_0^2}{1 - \frac{v^2}{c^2}}$$

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$$\text{or} \quad m^2 - \frac{m^2 v^2}{c^2} = m_0^2$$

$$\text{or} \quad m^2 c^2 - m^2 v^2 = m_0^2 c^2$$

$$\text{or} \quad m^2 c^2 = m^2 v^2 + m_0^2 c^2$$

Differentiating we get

$$2m\,dm\,c^2 = 2m\,dm\,v^2 + 2v.dv.m^2$$

$$c^2 dm = v^2 dm + mv\,dv \quad \text{——— (4)}$$

From eqn. (3) and (4) we have

$$dE_k = c^2 dm$$

If the mass of the body changes from m_0 to m and the corresponding change in kinetic energy E_k, we have on integration

$$\int dE_k = \int_{m_0}^{m} c^2 dm$$

$$\text{or} \quad E_k = c^2 (m - m_0) \quad \text{——— (5)}$$

The eqn (5) shows that the kinetic energy of a moving body is c^2 times the gain in mass. If thus indicates that the increase in mass is an indication and measure of the gain in kinetic energy.

The mass m_0 is often termed as the rest mass of the body and the term $m_0 c^2$ as the rest mass energy of the body. This rest mass energy is regarded as a form of internal energy inherent in the nature of particles constituting the matter. Rewriting eqn (5) we have

$$mc^2 = m_0 c^2 + E_k$$

or Total energy = Rest mass energy + kinetic energy

Denoting the total energy as E, we have

$$E = mc^2 \quad \text{——— (6)}$$

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It can be easily shown with the help of eqn (5) that if $v^2 \ll c$, the expression for E_K reduces to its classical form

$$E_K = c^2(m - m_0)$$

$$= c^2 m - c^2 m_0$$

$$= c^2 \frac{m_0}{\sqrt{1 - v^2/c^2}} - c^2 m_0$$

$$= c^2 m_0 \left\{ \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right\}$$

$$= c^2 m_0 \left\{ \left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right\}$$

$$= c^2 m_0 \left\{ 1 + \frac{1}{2}\frac{v^2}{c^2} - 1 \right\}$$

$$= m_0 c^2 + \frac{1}{2}\frac{v^2}{c^2}$$

$$= \frac{1}{2} m_0 v^2$$

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