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Gauss's divergence Theorem

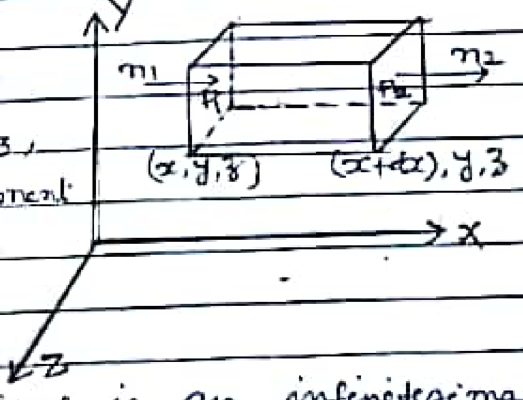
B-1 (H) Paper 1

Gauss's Theorem of divergence

The Theorem states that the normal surface integral of a ~~function~~ vector function $\vec{F}$ over a closed surface S (ie the flux across S) is equal to the volume integral of the divergence of the vector function over the volume V enclosed by the surface, ie

$$Flux = \iint_S \vec{F}.\vec{ds} = \iiint_V (div \vec{F})\, dv$$
$$= \iiint_V (\vec{\nabla}.\vec{F})\, dv$$

Let the surface S enclosing a volume V be divided up ~~into~~ into a very large number of elementary volumes in the form of cubes or ~~re~~ rectangular parallelopipeds adjoining each other. Let us imagine one such cube with its edges along the three Cartesian Co-ordinates and their lengths dx, dy and dz respectively.



The flux outwards through the left face $= -\int F_x\, dy dz$, where F_x is the x-component of the vector field $\vec{F}$ and integral extends over the area of the face.

Since the cube considered is an infinitesimal one, this integral may be taken to be very nearly equal to the product of ~~of~~ the x-component of $\vec{F}$ at the Centre P_1 of the face and the area dy dz of the face

So that, denoting the x-Component of $\vec{F}$ at the centre of the face by $F_x(P_1)$, we have

Flux through the left-face of the cube $= -F_x(P_1)\,dy\,dz$

Similarly, flux through the right-face of the cube

$$= F_x(P_2)\,dy\,dz,$$

the Component $F_x(P_2)$ now being positive

Taking $F_x(P_2) = F_x(P_1) + \frac{\partial F_x}{\partial x}dx$, we have

flux through the right face of the cube

$$= \left(F_x(P_1) + \frac{\partial F_x}{\partial x}dx\right)dy\,dz$$

$\therefore$ Flux through the left and right faces of the cube

$$= \left[F_x(P_1) + \frac{\partial F_x}{\partial x}dx\right]dy\,dz - F_x(P_1)\,dy\,dz$$

$$= \frac{\partial F_x}{\partial x}dx\,dy\,dz$$

Similarly, flux through top and bottom ~~of the~~ ~~Cube~~ faces of the cube

$$= \cancel{\frac{\partial F_x}{\partial x}}\ \frac{\partial F_y}{\partial y}dx\,dy\,dz$$

and flux through the remaining faces of the cube

$$= \frac{\partial F_z}{\partial z}dx\,dy\,dz$$

Therefore, flux through all the faces of the cube is given by the sum of all these. ~~So that~~ So that,

$$\iint_{\substack{\text{Surface}\\\text{of cube}}} \vec{F}\,ds = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)dx\,dy\,dz$$

where ds is the area of the surface of the cube.

Now $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \vec{\nabla}\cdot\vec{F}$ or $\operatorname{div}\vec{F}$

and $dx\,dy\,dz = dv$, volume of the cube

We, therefore, have $\iint_{\text{Surface of cube}} \vec{F}\,ds = (\vec{\nabla}\cdot\vec{F})\,dV$

Therefore, summing up the fluxes through all the elementary cubes

$$\iint_S \vec{E}\cdot ds = \iiint_V (\vec{\nabla}\cdot\vec{F})\,dV = \iiint_V (\operatorname{div}\vec{F})\,dV,$$

which is Gauss Theorem.